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# Concept of Experimental Accuracy and Simultaneous Measurements of Position and Momentum

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The concept of experimental accuracy is investigated in the context of the unbiased joint measurement processes defined by Arthurs and Kelly. A distinction is made between the errors of retrodiction and prediction. Four error–disturbance relationships are derived, analogous to the single error–disturbance relationship derived by Braginsky and Khalili in the context of single measurements of position only. A retrodictive and a predictive error–error relationship are also derived. The connection between these relationships and the extended uncertainty principle of Arthurs and Kelly is discussed. The similarities and differences between the quantum mechanical and classical concepts of experimental accuracy are explored. It is argued that these relationships provide grounds for questioning Uffink's conclusion that the concept of a simultaneous measurement of noncommuting observables is not fruitful.

## 1. INTRODUCTION

Notwithstanding the fundamental importance of the uncertainty principle, there is still, as Hilgevoord and Uffink (1990) have remarked, a great deal of discussion about what it actually says. The purpose of this paper is to add a few additional points to the discussion. We are particularly concerned with the idea that the uncertainty principle represents a constraint on the accuracy achievable in a simultaneous measurement of position and momentum.

The form of the uncertainty principle given in most modern textbooks is the inequality

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$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1}$$

where the quantities $\Delta x$, $\Delta p$ are defined in terms of the state of the system $|\psi\rangle$ by

$$\Delta x = \sqrt{\langle\psi|\hat{x}^2|\psi\rangle - \langle\psi|\hat{x}|\psi\rangle^2} \; \Delta p = \sqrt{\langle\psi|\hat{p}^2|\psi\rangle - \langle\psi|\hat{p}|\psi\rangle^2} \tag{2}$$

The first general proof of inequality (1) was actually given by Kennard (1927), not Heisenberg. We will accordingly refer to this form of the uncertainty principle as Kennard's inequality.

The proof of Kennard's inequality is based on the fact that $\langle p|\psi\rangle$ is the Fourier transform of $\langle x|\psi\rangle$. In his original paper, and again in his Chicago lectures, Heisenberg (1927, 1930) also gave another, quite different argument involving a $\gamma$-ray microscope. On the basis of this argument he interpreted $\Delta x$ and $\Delta p$ as experimental *errors* or *inaccuracies* . He thereby suggested that the uncertainty principle should be understood to mean, in the words of Bohm (1951), "If a measurement of position is made with accuracy $\Delta x$, and if a measurement of momentum is made *simultaneously* with accuracy $\Delta p$, then the product of the two errors can never be smaller than a number of order $\hbar$" (Bohm's emphasis). Heisenberg himself did not state the matter quite so plainly; however, one has the impression that he would have concurred with the above statement of Bohm's had it been put to him. The question arises: is this a valid interpretation of Kennard's inequality? The question has been discussed by Ballentine (1970), Wódkiewicz (1987), Hilgevoord and Uffink (1990), Raymer (1994), and de Muynck *et al.* (1994). We will here confine ourselves to remarking that the quantities $\Delta x$ and $\Delta p$ defined by Equations (2) cannot be interpreted as experimental errors in anything like the normal sense of the word because they only depend on the state $|\psi\rangle$. They are thus *intrinsic* properties of the isolated system. An experimental error, by contrast, ought to depend on the state of the measuring apparatus, as well as the state of the system. In other words, it should partly depend on quantities which are *extrinsic* to the system.

Suppose that in Heisenberg's microscope *gedanken* experiment, one were to make the microscope go out of focus. This should have the effect of increasing the error in the measurement of $x$. But it will have no effect on the quantity $\Delta x$, since this only depends on the initial state of the particle whose position is being measured.

These considerations do not mean that the statement of Bohm's quoted above is incorrect. They do, however, mean that it is not a consequence of the inequality proved by Kennard. Rather, it represents (if true) an independent physical principle. For the sake of distinctness let us give it a name: the error principle.

The problem we now face is that whereas there exists a rigorous mathematical proof of Kennard's inequality, the status of the error principle is much more ambiguous. Indeed, the very meaning of the concepts involved—the concept of a simultaneous measurement of position and momentum, and the concept of experimental accuracy—continues to be the subject of discussion.

One approach to the problem is that based on the concept of a "fuzzy" or "stochastic" measurement, due to Prugovecki (1984), Holevo (1982), Busch and Lahti (1984), Martens and de Muynck (1992), de Muynck *et al.* (1994), and others [for additional references see the works just cited and Uffink (1994)]. Uffink (1994) has identified a number of objections to this approach. His conclusion is "that the claim that within this formalism a joint unsharp measurement of position and momentum . . . is possible is false". Moreover, he doubts whether matters could be remedied by adopting a different approach. He considers that "the formalism of quantum theory, as presented by von Neumann, simply has no room for a description of a point measurement of position and momentum at all"—not even a less than perfectly accurate joint measurement.

We acknowledge the force of Uffink's arguments. Nevertheless, we are unwilling to accept his analysis as the last word on the subject. In the first place, ordinary laboratory practice depends on the assumption that it is possible to make simultaneous, imperfectly accurate determinations of the position and momentum of macroscopic objects. If it is true that quantum mechanics does not allow for the existence of such measurements, then one of two things would seem to follow: either normal laboratory practice is based on a misconception, in which case much of the evidential basis for modern physics (including quantum mechanics) would simply collapse; or else quantum mechanics does not apply on the macroscopic scale. In short, Uffink's conclusion has some fairly momentous consequences. This is not, of course, a reason for rejecting Uffink's conclusion. It is, however, a reason for reexamining the question, to see if there is some way of avoiding his conclusion.

In the second place, a number of authors (Arthurs and Kelly, 1965; Braunstein *et al.*, 1991; Stenholm, 1992; Leonhardt and Paul, 1993; Törma *et al.*, 1995) have described several specific processes which might be described (and which they do describe) as simultaneous measurements of position and momentum. Their work is logically independent of the work criticized by Uffink, and it is therefore not open to the same objections. Indeed, Uffink explicitly states that he does not mean to impugn the approach of these authors (although he does question whether it is "fruitful" to interpret the processes they describe as simultaneous measurements of noncommuting observables).

Within the context of their approach Arthurs and Kelly (1965) have derived an "extended" or "generalized" uncertainty principle (also see Wódkiewicz, 1987; Arthurs and Goodman, 1988; Raymer, 1994; Leonhardt and Paul, 1995). Let $\Delta\mu_{\mathrm{Xf}}$ (respectively $\Delta\mu_{\mathrm{Pf}}$) be the standard deviation for the outcome of the measurement of $\hat{x}$ (respectively $\hat{p}$). Then, subject to certain restrictive assumptions regarding the nature of the measurement process, Arthurs and Kelly show

$$\Delta\mu_{\mathrm{Xf}}\, \Delta\mu_{\mathrm{Pf}} \geq \hbar \tag{3}$$

where we have employed a different notation from that of Arthurs and Kelly (the reasons for this notation will become clear in the next section).

The quantities $\Delta\mu_{\mathrm{Xf}}$ and $\Delta\mu_{\mathrm{Pf}}$ are not interpretable as experimental errors. However, they do depend on the initial state of the apparatus, as well as the initial state of the system. Moreover, the increase in the lower bound set by inequality (3) as compared with Kennard's inequality can be taken as a quantitative indication of the noise introduced by the measurement. So, although this relation cannot be regarded as a quantitative expression of the error principle (the statement of Bohm's quoted above), it may at least be regarded as a step in that direction.

Another relation relevant to our problem is the one derived by Braginsky and Khalili (1992), in the context of single measurements of position only. Braginsky and Khalili define a quantity $\Delta x_{\mathrm{measure}}$, representing the error in the measurement of $\hat{x}$, and a quantity $\Delta p_{\mathrm{perturbation}}$, representing the disturbance of the conjugate quantity $\hat{p}$, and they show

$$\Delta x_{\mathrm{measure}} \Delta p_{\mathrm{pertubation}} \geq \frac{\hbar}{2} \tag{4}$$

provided that the measurement is of the special kind which they describe as linear. Their inequality does not refer to simultaneous measurements of position *and* momentum, and only one of the two quantities on the left-hand side is interpretable as an experimental error. However, its existence encourages us to believe that a similar approach might prove fruitful in the problem of interest here.

The purpose of this paper is to combine and to develop the approaches of Arthurs and Kelly on one hand and of Braginsky and Khalili on the other in an attempt to find a precise, quantitative expression of the error principle as stated by Bohm in the passage quoted above.

The result of our analysis is to show that there are in fact two different error principles, corresponding to the predictive and retrodictive aspects of a measurement process as discussed by Hilgevoord and Uffink (1990) (also see Prugovecki, 1973, 1975). In addition, we derive four error–disturbance relationships (in place of the single relationship derived by Braginsky and Khalili).

The six inequalities which we derive in the following sections, together with Kennard's inequality, gives a total of seven inequalities. If our analysis is correct all of these inequalities are needed to capture the full intuitive content of Heisenberg's original paper.

## 2. THE ARTHURS–KELLY PROCESS

We begin by considering a specific example of a simultaneous measurement process, namely the process described by Arthurs and Kelly (1965) (also see Braunstein *et al.*, 1991; Stenholm, 1992). Suppose that we have a system interacting with a measuring apparatus, or meter. The system has one degree of freedom, with position $\hat{x}$ and conjugate momentum $\hat{p}$. The measuring apparatus has two degrees of freedom, comprising two pointer observables $\hat{\mu}_X$, $\hat{\mu}_P$ with conjugate momenta $\hat{\pi}_X$, $\hat{\pi}_P$. The pointer observables $\hat{\mu}_X$, $\hat{\mu}_P$ give the result of the measurement. We have the commutation relations

$$[\hat{x}, \hat{p}] = [\hat{\mu}_X, \hat{\pi}_X] = [\hat{\mu}_P, \hat{\pi}_P] = i\hbar$$

these being the only nonvanishing commutators between the six operators $\hat{x}$, $\hat{p}$, $\hat{\mu}_X$, $\hat{\pi}_X$, $\hat{\mu}_P$, $\hat{\pi}_P$.

The unitary evolution operator describing the measurement interaction is

$$\hat{U} = \exp\left[-\frac{i}{\hbar}(\hat{\pi}_P\hat{p} + \hat{\pi}_X\hat{x})\right]$$

Suppose that the system + apparatus composite is initially in the product state $|\psi \otimes \phi_{ap}\rangle$, where $|\psi\rangle$ is the initial state of the system and $|\phi_{ap}\rangle$ is the initial state of the apparatus. The probability distribution for the result of the measurement is then given by

$$\rho(\mu_X, \mu_P) = \int dx |\langle x, \mu_X, \mu_P| \hat{U} |\psi \otimes \phi_{ap}\rangle|^2$$

In order to describe the experimental errors, and the disturbance of the system by the measurement process, it is convenient to switch to the Heisenberg picture. Let $\hat{\mathbb{O}}$ be any of the operators $\hat{x}$, $\hat{p}$, $\hat{\mu}_X$, $\hat{\pi}_X$, $\hat{\mu}_P$, $\hat{\pi}_P$. We then define the initial Heisenberg picture operator $\hat{\mathbb{O}}_i$ and final Heisenberg picture operator $O_f$ by

$$\hat{\mathbb{O}}_i = \hat{\mathbb{O}}$$

$$\hat{\mathbb{O}}_f = \hat{U}^\dagger \hat{\mathbb{O}} \hat{U}$$

It is readily found that

$$
\begin{aligned}
\hat{x}_{\mathrm{f}} &= \hat{U}^{\dagger}\hat{x}\hat{U} = \hat{x} + \hat{\pi}_{\mathrm{P}} \\
\hat{p}_{\mathrm{f}} &= \hat{U}^{\dagger}\hat{p}\hat{U} = \hat{p} - \hat{\pi}_{\mathrm{X}} \\
\hat{\mu}_{\mathrm{Xf}} &= \hat{U}^{\dagger}\hat{\mu}_{\mathrm{X}}\hat{U} = \hat{\mu}_{\mathrm{X}} + \hat{x} + \tfrac{1}{2}\,\hat{\pi}_{\mathrm{P}} \\
\hat{\mu}_{\mathrm{Pf}} &= \hat{U}^{\dagger}\hat{\mu}_{\mathrm{P}}\hat{U} = \hat{\mu}_{\mathrm{P}} + \hat{p} - \tfrac{1}{2}\,\hat{\pi}_{\mathrm{X}} \\
\hat{\pi}_{\mathrm{Xf}} &= \hat{U}^{\dagger}\hat{\pi}_{\mathrm{X}}\hat{U} = \hat{\pi}_{\mathrm{X}} \\
\hat{\pi}_{\mathrm{Pf}} &= \hat{U}^{\dagger}\hat{\pi}_{\mathrm{P}}\hat{U} = \hat{\pi}_{\mathrm{P}}
\end{aligned}
\tag{5}
$$

We now define the retrodictive error operators

$$
\begin{aligned}
\hat{\varepsilon}_{\mathrm{Xi}} &= \hat{\mu}_{\mathrm{Xf}} - \hat{x}_{\mathrm{i}} \\
\hat{\varepsilon}_{\mathrm{Pi}} &= \hat{\mu}_{\mathrm{Pf}} - \hat{p}_{\mathrm{i}}
\end{aligned}
\tag{6}
$$

the predictive error operators

$$
\begin{aligned}
\hat{\varepsilon}_{\mathrm{Xf}} &= \hat{\mu}_{\mathrm{Xf}} - \hat{x}_{\mathrm{f}} \\
\hat{\varepsilon}_{\mathrm{Pf}} &= \hat{\mu}_{\mathrm{Pf}} - \hat{p}_{\mathrm{f}}
\end{aligned}
\tag{7}
$$

and the disturbance operators

$$
\begin{aligned}
\hat{\delta}_{\mathrm{X}} &= \hat{x}_{\mathrm{f}} - \hat{x}_{\mathrm{i}} \\
\hat{\delta}_{\mathrm{P}} &= \hat{p}_{\mathrm{f}} - \hat{p}_{\mathrm{i}}
\end{aligned}
\tag{8}
$$

The motivation for these definitions will be clearest if we think, for a moment, in classical terms. In that case $\hat{\varepsilon}_{\mathrm{Xi}}$, $\hat{\varepsilon}_{\mathrm{Pi}}$ give the difference between the final pointer positions and the initial system observables $\hat{x}_{\mathrm{i}}$, $\hat{p}_{\mathrm{i}}$. In other words they tell us how accurately the result of the measurement reflects the initial state of the system, *before* the measurement was carried out, which is why we refer to them as retrodictive error operators. On the other hand $\hat{\varepsilon}_{\mathrm{Xf}}$, $\hat{\varepsilon}_{\mathrm{Pf}}$ give the difference between the final pointer positions and the final system observables $\hat{x}_{\mathrm{f}}$, $\hat{p}_{\mathrm{f}}$. They therefore tell us how accurately the result of the measurement reflects the final state of the system, *after* the measurement has been completed, which is why we refer to them as predictive error operators. Lastly, $\hat{\delta}_{\mathrm{X}}$, $\hat{\delta}_{\mathrm{P}}$ give the difference between the final system observables $\hat{x}_{\mathrm{f}}$, $\hat{p}_{\mathrm{f}}$ and the initial system observables $\hat{x}_{\mathrm{i}}$, $\hat{p}_{\mathrm{i}}$. They therefore describe the disturbance of the system by the measurement process.

Of course, we are actually talking about quantum mechanics, not classical mechanics. Our definitions therefore raise some important conceptual questions. We do not wish to minimize these questions. We do, however, wish to defer discussing them until after we have derived some quantitative formulas. It is to be observed that whatever the precise conceptual or philosophical status of the quantities just introduced, they are well defined mathematically.

In order to obtain numerical indications of the accuracy and disturbance we take the rms values of the operators just defined. We thus have the rms errors of retrodiction

$$\begin{aligned}\Delta_{\mathrm{ei}}x &= \sqrt{\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Xi}}^2|\psi \otimes \phi_{\mathrm{ap}}\rangle}\\ \Delta_{\mathrm{ei}}p &= \sqrt{\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Pi}}^2|\psi \otimes \phi_{\mathrm{ap}}\rangle}\end{aligned} \tag{9}$$

the rms errors of prediction

$$\begin{aligned}\Delta_{\mathrm{ef}}x &= \sqrt{\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Xf}}^2|\psi \otimes \phi_{\mathrm{ap}}\rangle}\\ \Delta_{\mathrm{ef}}p &= \sqrt{\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Pf}}^2|\psi \otimes \phi_{\mathrm{ap}}\rangle}\end{aligned} \tag{10}$$

and the rms disturbances

$$\begin{aligned}\Delta_{\mathrm{d}}x &= \sqrt{\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\delta}_{\mathrm{X}}^2|\psi \otimes \phi_{\mathrm{ap}}\rangle}\\ \Delta_{\mathrm{d}}p &= \sqrt{\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\delta}_{\mathrm{P}}^2|\psi \otimes \phi_{\mathrm{ap}}\rangle}\end{aligned} \tag{11}$$

The above definitions apply to any measurement process. Let us now specialize to the case of the Arthurs–Kelly process. Inserting (5) in the defining equations (6)–(8) gives

$$\begin{aligned}\hat{\varepsilon}_{\mathrm{Xi}} &= \hat{\mu}_{\mathrm{X}} + \tfrac{1}{2}\hat{\pi}_{\mathrm{P}}, &\quad \hat{\varepsilon}_{\mathrm{Xf}} &= \hat{\mu}_{\mathrm{X}} - \tfrac{1}{2}\hat{\pi}_{\mathrm{P}}, &\quad \hat{\delta}_{\mathrm{X}} &= \hat{\pi}_{\mathrm{P}}\\ \hat{\varepsilon}_{\mathrm{Pi}} &= \hat{\mu}_{\mathrm{P}} - \tfrac{1}{2}\hat{\pi}_{\mathrm{X}}, &\quad \hat{\varepsilon}_{\mathrm{Pf}} &= \hat{\mu}_{\mathrm{P}} + \tfrac{1}{2}\hat{\pi}_{\mathrm{X}}, &\quad \hat{\delta}_{\mathrm{P}} &= -\hat{\pi}_{\mathrm{X}}\end{aligned} \tag{12}$$

It is to be observed that the error and disturbance operators only depend on the pointer positions and momenta. It follows that the rms errors and disturbances as defined by equations (9)–(11) are independent of the initial system state. This is, of course, a peculiarity of the Arthurs–Kelly process. We do not expect it to be true generally.

Using equations (12), we find

$$\begin{aligned}[\hat{\varepsilon}_{\mathrm{Xi}}, \hat{\varepsilon}_{\mathrm{Pi}}] &= -i\hbar, &\quad [\hat{\varepsilon}_{\mathrm{Xi}}, \hat{\delta}_{\mathrm{P}}] &= -i\hbar, &\quad [\hat{\delta}_{\mathrm{X}}, \hat{\varepsilon}_{\mathrm{Pi}}] &= -i\hbar\\ [\hat{\varepsilon}_{\mathrm{Xf}}, \hat{\varepsilon}_{\mathrm{Pf}}] &= i\hbar, &\quad [\hat{\varepsilon}_{\mathrm{Xf}}, \hat{\delta}_{\mathrm{P}}] &= -i\hbar, &\quad [\hat{\delta}_{\mathrm{X}}, \hat{\varepsilon}_{\mathrm{Pf}}] &= -i\hbar\end{aligned} \tag{13}$$

these being the only nonvanishing commutation relationships between members of the set $\hat{\varepsilon}_{\mathrm{Xi}}$, $\hat{\varepsilon}_{\mathrm{Pi}}$, $\hat{\varepsilon}_{\mathrm{Xf}}$, $\hat{\varepsilon}_{\mathrm{Pf}}$, $\hat{\delta}_{\mathrm{X}}$, $\hat{\delta}_{\mathrm{P}}$. Taking this result in conjunction with the defining equations (9)–(11), we deduce, a retrodictive error relationship

$$\Delta_{\mathrm{ei}}x\Delta_{\mathrm{ei}}p \geq \frac{\hbar}{2} \tag{14}$$

a predictive error relationship

$$\Delta_{\mathrm{ef}}x\Delta_{\mathrm{ef}}p \geq \frac{\hbar}{2} \tag{15}$$

and four error–disturbance relationships

$$\Delta_{\mathrm{ei}}x\Delta_{\mathrm{d}}p \geq \frac{\hbar}{2}, \qquad \Delta_{\mathrm{ei}}p\Delta_{\mathrm{d}}x \geq \frac{\hbar}{2}, \qquad \Delta_{\mathrm{ef}}x\Delta_{\mathrm{d}}p \geq \frac{\hbar}{2}, \qquad \Delta_{\mathrm{ef}}p\Delta_{\mathrm{d}}x \geq \frac{\hbar}{2} \tag{16}$$

Equations (14) and (15) together constitute a quantitative expression of the semiintuitive error principle, as stated by Bohm (1951) in the passage quoted earlier. Equations (16) provide a quantitative expression of the principle that an increased degree of accuracy in the measurement of one observable can only be achieved at the expense of an increased degree of disturbance in the canonically conjugate observable.

The reason one needs two inequalities to capture the full content of the error principle is the fact that one has to distinguish the errors of prediction from the errors of retrodiction. In classical physics it is not usually necessary to emphasize this distinction. This is because in classical physics the disturbance of the system by the measurement can in principle be made negligible. In quantum mechanics, however, the backreaction of the apparatus on the system is very important. As a result, the distinction between the two kinds of error is also essential. In fact, it is an immediate consequence of the definitions that

$$\hat{\delta}_{\mathrm{X}} = \hat{\varepsilon}_{\mathrm{Xi}} - \hat{\varepsilon}_{\mathrm{Xf}}$$

$$\hat{\delta}_{\mathrm{P}} = \hat{\varepsilon}_{\mathrm{Pi}} - \hat{\varepsilon}_{\mathrm{Pf}}$$

It follows that if the disturbances cannot be assumed to be negligible, then neither can the difference between the retrodictive and predictive errors.

The reason that there are four error-disturbance relations in our analysis, but only one in the analysis of Braginsky and Khalili is first, that Braginsky and Khalili do not consider simultaneous measurements of $\hat{x}$ and $\hat{p}$ and second, that they only consider the error of retrodiction (as we have termed it).

Arthurs and Kelly consider an initial apparatus state with wave function of the form

$$\langle \mu_{\mathrm{X}}, \mu_{\mathrm{P}} | \phi_{\mathrm{ap}} \rangle = \frac{2}{\sqrt{h}} \exp\left( -\frac{1}{\lambda^2} \mu_{\mathrm{X}}^2 - \frac{\lambda^2}{h^2} \mu_{\mathrm{P}}^2 \right)$$

The reader may easily verify that for this choice of $|\phi_{\mathrm{ap}}\rangle$ the errors are given by

$$\Delta_{\mathrm{ei}}x = \Delta_{\mathrm{ef}}x = \frac{\lambda}{\sqrt{2}}$$

$$\Delta_{\mathrm{ei}}p = \Delta_{\mathrm{ef}}p = \frac{\hbar}{\sqrt{2}\lambda}$$

We see that the apparatus states considered by Arthurs and Kelly minimize both the product $\Delta_{ei} x \, \Delta_{ei} p$, and the product $\Delta_{ef} x \, \Delta_{ef} p$. In other words, they maximize both the retrodictive and the predictive accuracy of the measurement.

## 3. UNBIASED MEASUREMENTS

After introducing the particular process which we discussed in the last section, Arthurs and Kelly (1965) go on to define a general class of measurement processes. They show that their extended uncertainty principle, relation (3) above, holds for every process in this class (also see Arthurs and Goodman, 1988). It is natural to ask whether the error–error and error-disturbance relations (14)–(16) also generalize.

As before, the system is assumed to interact with a measuring apparatus characterized by two pointer observables $\hat{\mu}_X$, $\hat{\mu}_P$ which commute with each other and with the observables being measured $\hat{x}$, $\hat{p}$. However, the apparatus may now have additional degrees of freedom apart from these two.

Let $\hat{U}$ be the unitary evolution operator describing the measurement interaction, and define error and disturbance operators as in the last section. Arthurs and Kelly assume that the evolution operator $\hat{U}$ and initial apparatus state $|\phi_{ap}\rangle$ are such that

$$\begin{aligned} \langle \psi \otimes \phi_{ap} | \hat{\mu}_{Xf} | \psi \otimes \phi_{ap} \rangle &= \langle \psi \otimes \phi_{ap} | \hat{x}_i | \psi \otimes \phi_{ap} \rangle \\ \langle \psi \otimes \phi_{ap} | \hat{\mu}_{Pf} | \psi \otimes \phi_{ap} \rangle &= \langle \psi \otimes \phi_{ap} | \hat{p}_i | \psi \otimes \phi_{ap} \rangle \end{aligned} \tag{17}$$

uniformly, for every initial system state $|\psi\rangle$. In our terminology this condition amounts to the requirement that there be no systematic errors of retrodiction:

$$\begin{aligned} \langle \psi \otimes \phi_{ap} | \hat{\varepsilon}_{Xi} | \psi \otimes \phi_{ap} \rangle &= 0 \\ \langle \psi \otimes \phi_{ap} | \hat{\varepsilon}_{Pi} | \psi \otimes \phi_{ap} \rangle &= 0 \end{aligned} \tag{18}$$

for all $|\psi\rangle$. We will accordingly refer to such a measurement as *retrodictively unbiased*.

It is natural also to impose the requirement that the measurement be *predictively unbiased*:

$$\begin{aligned} \langle \psi \otimes \phi_{ap} | \hat{\varepsilon}_{Xf} | \psi \otimes \phi_{ap} \rangle &= 0 \\ \langle \psi \otimes \phi_{ap} | \hat{\varepsilon}_{Pf} | \psi \otimes \phi_{ap} \rangle &= 0 \end{aligned}$$

for all $|\psi\rangle$.

We now show that all six of the error–error and error–disturbance relations (14)–(16) continue to hold for every measurement which is both retrodictively and predictively unbiased. We will do so by using a method similar to the one used by Arthurs and Kelly to prove their extended uncertainty principle (3).

We begin with the predictive error relationship. We have

$$[\hat{x}_{\mathrm{f}}, \hat{p}_{\mathrm{f}}] = \hat{U}^{\dagger}[\hat{x}, \hat{p}]\, \hat{U} = i\hbar$$

This is the only nonvanishing commutator between members of the set $\hat{x}_{\mathrm{f}}$, $\hat{p}_{\mathrm{f}}$, $\hat{\mu}_{\mathrm{Xf}}$, $\hat{\mu}_{\mathrm{pf}}$. Therefore

$$[\hat{\varepsilon}_{\mathrm{Xf}}, \hat{\varepsilon}_{\mathrm{Pf}}] = [(\hat{\mu}_{\mathrm{Xf}} - \hat{x}_{\mathrm{f}}),(\hat{\mu}_{\mathrm{Pf}} - \hat{p}_{\mathrm{f}})] = i\hbar$$

We deduce

$$\Delta_{\mathrm{ef}}x\, \Delta_{\mathrm{ef}}p \geq \frac{\hbar}{2}$$

We made no use of the assumption that the measurement is unbiased in deriving this inequality. The predictive error relationship therefore holds quite generally. The remaining relationships mix Heisenberg picture operators defined at different times, and for these we must work a little harder.

Given an initial system state $|\psi\rangle$, let $|\psi'\rangle = \hat{x}_i|\psi\rangle$. If the measurement is retrodictively unbiased, we then have, from the proposition proved in the Appendix,

$$\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Xi}}\hat{x}_{\mathrm{i}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = \langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Xi}}|\, \psi' \otimes \phi_{\mathrm{ap}}\rangle = 0 \tag{19}$$

Similarly

$$\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{x}_{\mathrm{i}}\hat{\varepsilon}_{\mathrm{Xi}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = 0 \tag{20}$$

and

$$\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Xi}}\hat{p}_{\mathrm{i}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = \langle\psi \otimes \phi_{\mathrm{ap}}|\hat{p}_{\mathrm{i}}\hat{\varepsilon}_{\mathrm{Xi}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = 0$$
$$\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Pi}}\hat{x}_{\mathrm{i}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = \langle\psi \otimes \phi_{\mathrm{ap}}|\hat{x}_{\mathrm{i}}\hat{\varepsilon}_{\mathrm{Pi}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = 0$$
$$\langle\psi \otimes \phi_{\mathrm{ap}}|\hat{\varepsilon}_{\mathrm{Pi}}\hat{p}_{\mathrm{i}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = \langle\psi \otimes \phi_{\mathrm{ap}}|\hat{p}_{\mathrm{i}}\hat{\varepsilon}_{\mathrm{Pi}}|\, \psi \otimes \phi_{\mathrm{ap}}\rangle = 0$$

Using these equations and the definitions of $\hat{\varepsilon}_{\mathrm{Xi}}$, $\hat{\varepsilon}_{\mathrm{Pi}}$, it is readily inferred that

$$\begin{aligned} &\langle\psi \otimes \phi_{\mathrm{ap}}|\, [\hat{x}_{\mathrm{i}}, \hat{\mu}_{\mathrm{Xf}}]|\psi \otimes \phi_{\mathrm{ap}}\rangle = 0, \qquad &&\langle\psi \otimes \phi_{\mathrm{ap}}|[\hat{\mu}_{\mathrm{Xf}}, \hat{p}_{\mathrm{i}}]|\psi \otimes \phi_{\mathrm{ap}}\rangle = i\hbar \\ &\langle\psi \otimes \phi_{\mathrm{ap}}|[\hat{x}_{\mathrm{i}}, \hat{\mu}_{\mathrm{Pf}}]|\psi \otimes \phi_{\mathrm{ap}}\rangle = i\hbar, \qquad &&\langle\psi \otimes \phi_{\mathrm{ap}}|[\hat{\mu}_{\mathrm{Pf}}, \hat{p}_{\mathrm{i}}]|\psi \otimes \phi_{\mathrm{ap}}\rangle = 0 \end{aligned} \tag{21}$$

which, together with the fact that $\hat{\mu}_{\mathrm{Xf}}$ and $\hat{\mu}_{\mathrm{Pf}}$ commute, implies

$$\begin{aligned} \langle\psi \otimes \phi_{\mathrm{ap}}|[\hat{\varepsilon}_{\mathrm{Xi}}, \hat{\varepsilon}_{\mathrm{Pi}}]|\psi \otimes \phi_{\mathrm{ap}}\rangle &= \langle\psi \otimes \phi_{\mathrm{ap}}|[(\hat{\mu}_{\mathrm{Xf}} - \hat{x}_{\mathrm{i}}), (\hat{\mu}_{\mathrm{Pf}} - \hat{p}_{\mathrm{i}})]|\psi \otimes \phi_{\mathrm{ap}}\rangle \\ &= -i\hbar \end{aligned}$$

for all $|\psi\rangle$. Consequently

$$\Delta_{\mathrm{ei}} x \Delta_{\mathrm{ei}} p \geq \frac{\hbar}{2} \tag{22}$$

In proving this inequality, we only used the assumption that the measurement is retrodictively unbiased. The retrodictive error relationship is therefore valid under the same set of assumptions which Arthurs and Kelly make in order to prove their extended uncertainty principle.

Suppose, now, that the measurement is both retrodictively and predictively unbiased. Then, by an argument similar to that used in proving equations (21), we find

$$\begin{aligned}
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{x}_{\mathrm{i}}, \hat{x}_{\mathrm{f}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle &= 0, \\
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{x}_{\mathrm{f}}, \hat{p}_{\mathrm{i}}] [\psi \otimes \phi_{\mathrm{ap}} \rangle &= i\hbar \\
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{x}_{\mathrm{i}}, \hat{p}_{\mathrm{f}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle &= i\hbar, \\
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{p}_{\mathrm{f}}, \hat{p}_{\mathrm{i}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle &= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{\varepsilon}_{\mathrm{xi}}, \hat{\delta}_{\mathrm{p}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle &= \langle \psi \otimes \phi_{\mathrm{ap}} | [(\hat{\mu}_{\mathrm{xf}} - \hat{x}_{\mathrm{i}}), (\hat{p}_{\mathrm{f}} - \hat{p}_{\mathrm{i}})] | \psi \otimes \phi_{\mathrm{ap}} \rangle \\
&= -i\hbar
\end{aligned}$$

Similarly

$$\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{\varepsilon}_{\mathrm{Xf}}, \hat{\delta}_{\mathrm{P}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle = -i\hbar$$

and

$$\begin{aligned}
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{\varepsilon}_{\mathrm{Pi}}, \hat{\delta}_{\mathrm{X}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle &= i\hbar \\
\langle \psi \otimes \phi_{\mathrm{ap}} | [\hat{\varepsilon}_{\mathrm{Pf}}, \hat{\delta}_{\mathrm{X}}] | \psi \otimes \phi_{\mathrm{ap}} \rangle &= i\hbar
\end{aligned}$$

Hence

$$\Delta_{\mathrm{ei}} x \Delta_{\mathrm{d}} p \geq \frac{\hbar}{2}, \quad \Delta_{\mathrm{ei}} p \Delta_{\mathrm{d}} x \geq \frac{\hbar}{2}, \quad \Delta_{\mathrm{ef}} x \Delta_{\mathrm{d}} p \geq \frac{\hbar}{2}, \quad \Delta_{\mathrm{ef}} p \Delta_{\mathrm{d}} x \geq \frac{\hbar}{2} \tag{23}$$

It would be interesting to see if one can remove the restriction to measurement processes which are retrodictively unbiased [in the case of inequality (22)], or retrodictively and predictively unbiased [in the case of inequalities (23)].

## 4. THE ARTHURS–KELLY PRINCIPLE AND RELATED INEQUALITIES

For the sake of completeness we briefly indicate the connection between the inequalities proved in the last section and the extended uncertainty principle of Arthurs and Kelly (1965).

Suppose that the measurement is retrodictively unbiased. Then

$$\langle\psi \otimes \phi_{\rm ap}|\hat{\mu}_{\rm Xf}|\psi \otimes \phi_{\rm ap}\rangle = \langle\psi \otimes \phi_{\rm ap}|\hat{x}_{\rm i}|\psi \otimes \phi_{\rm ap}\rangle$$

In view of equations (19) and (20) we also have

$$\begin{aligned}\langle\psi \otimes \phi_{\rm ap}|\hat{\mu}_{\rm Xf}^2|\psi \otimes \phi_{\rm ap}\rangle &= \langle\psi \otimes \phi_{\rm ap}|(\hat{x}_{\rm i} + \hat{\varepsilon}_{\rm Xi})^2|\psi \otimes \phi_{\rm ap}\rangle \\ &= \langle\psi \otimes \phi_{\rm ap}|\hat{x}_{\rm i}^2|\psi \otimes \phi_{\rm ap}\rangle + \langle\psi \otimes \phi_{\rm ap}|\hat{\varepsilon}_{\rm Xi}^2|\psi \otimes \phi_{\rm ap}\rangle\end{aligned}$$

Using equations (17) and (18), we deduce

$$(\Delta\mu_{\rm Xf})^2 = (\Delta x_{\rm i})^2 + (\Delta_{\rm ei}x)^2 \tag{24}$$

where $\Delta\mu_{\rm Xf}$, $\Delta x_{\rm i}$ represent uncertainties calculated in the usual way, according to the prescription of equation (2). Similarly

$$(\Delta\mu_{\rm Pf})^2 = (\Delta p_{\rm i})^2 + (\Delta_{\rm ei}p)^2 \tag{25}$$

We see that $\Delta_{\rm ei}x$ and $\Delta_{\rm ei}p$ determine the increases in the variances of the distribution of results, as compared with the intrinsic variances of the initial system state.

Equations (24) and (25), together with Kennard's inequality (1) and the retrodictive error relationship (22), imply

$$\begin{aligned}(\Delta\mu_{\rm Xf})^2(\Delta\mu_{\rm Pf})^2 &= ((\Delta x_{\rm i})^2 + (\Delta_{\rm ei}x)^2)((\Delta p_{\rm i})^2 + (\Delta_{\rm ei}p)^2) \\ &\geq \frac{\hbar^2}{4}((\Delta x_{\rm i})^2 + (\Delta_{\rm ei}x)^2)\left(\frac{1}{(\Delta x_{\rm i})^2} + \frac{1}{(\Delta_{\rm ei}x)^2}\right) \\ &= \frac{\hbar^2}{4}\left(2 + \frac{(\Delta x_{\rm i})^2}{(\Delta_{\rm ei}x)^2} + \frac{(\Delta_{\rm ei}x)^2}{(\Delta x_{\rm i})^2}\right) \\ &\geq \hbar^2\end{aligned} \tag{26}$$

which is the extended principle of Arthurs and Kelly.

If the measurement is both retrodictively and predictively unbiased we can also prove, by a similar argument,

$$\begin{aligned}(\Delta x_{\rm f})^2 &= (\Delta x_{\rm i})^2 + (\Delta_{\rm d}x)^2 \\ (\Delta P_{\rm f})^2 &= (\Delta p_{\rm i})^2 + (\Delta_{\rm d}p)^2\end{aligned} \tag{27}$$

showing how the mean square disturbances determine the extent of the increase in the system state variances. These inequalities do not imply an increase in the lower bound on the product $\Delta x_{\rm f}\Delta p_{\rm f}$ because the disturbances can both be made arbitrarily small (at the expense of making the measurement very inaccurate). The lower bound for the final system state uncertainties is therefore the same as that for the initial state ones, namely

$$\Delta x_{\mathrm{f}}\, \Delta p_{\mathrm{f}} \geq \frac{\hbar}{2}$$

On the other hand, the lower bound on the products $\Delta x_{\mathrm{f}}\, \Delta\mu_{\mathrm{Pf}}$ and $\Delta\mu_{\mathrm{Xf}}\, \Delta p_{\mathrm{f}}$ is larger than the one set by Kennard's inequality. In fact, (24), (25), and (27), together with the error-disturbance relations (23) are readily seen to imply

$$\Delta x_{\mathrm{f}}\, \Delta\mu_{\mathrm{Pf}} \geq \hbar$$

$$\Delta\mu_{\mathrm{Xf}}\, \Delta p_{\mathrm{t}} \geq \hbar$$

## 5. THE QUESTION OF INTERPRETATION

We now come to the question which we have been ignoring up to now. We have been referring to the quantities $\Delta_{\mathrm{ei}} x$, $\Delta_{\mathrm{ei}} p$, $\Delta_{\mathrm{ef}} x$, $\Delta_{\mathrm{ef}} p$ as experimental errors and the quantities $\Delta_{\mathrm{d}} x$, $\Delta_{\mathrm{d}} p$ as disturbances. Is this terminology really justified?

Let us begin with the quantities $\Delta_{\mathrm{ef}} x$, $\Delta_{\mathrm{ef}} p$. The observables $\hat{\mu}_{\mathrm{Xf}}$, $\hat{\mu}_{\mathrm{Pf}}$, $\hat{x}_{\mathrm{f}}$ commute, and can therefore be simultaneously determined with arbitrarily high precision. Alternatively, one may determine the values of $\hat{\mu}_{\mathrm{Xf}}$, $\hat{\mu}_{\mathrm{Pf}}$ without perturbing $\hat{x}_{\mathrm{f}}$. We may therefore envisage a procedure, in which one *first* makes a highly accurate determination of the meter readings and then checks the value of $\hat{\mu}_{\mathrm{Xf}}$ by making an (immediately) *subsequent* highly accurate determination of $\hat{x}_{\mathrm{f}}$. Suppose that one takes numerous copies of the system, all prepared in the same state, performs this procedure on each of them, and calculates the rms value of the differences $\mu_{\mathrm{Xf}} - x_{\mathrm{f}}$. Then, provided that the verification of $\hat{x}_{\mathrm{f}}$ is carried out immediately after the determination of $\hat{\mu}_{\mathrm{Xf}}$, $\hat{\mu}_{\mathrm{Pf}}$, the quantity which results will almost certainly be no larger than an amount $\sim\Delta_{\mathrm{ef}} x$.

We can equally well envisage a procedure in which one makes a second, verificatory measurement of $\hat{p}_{\mathrm{f}}$ immediately after recording the meter readings. If one repeated this procedure many times, then the rms value of the differences $\mu_{\mathrm{Pf}} - p_{\mathrm{f}}$ would almost certainly be no larger than an amount $\sim\Delta_{\mathrm{ef}} p$.

Suppose, now, that one has recorded the meter readings to be $\mu_{\mathrm{Xf}}$, $\mu_{\mathrm{Pf}}$. What can be deduced about the likely state of the system? It is, of course, impossible to check the values of both $\hat{x}_{\mathrm{f}}$ and $\hat{p}_{\mathrm{f}}$ to arbitrarily high precision. It is, however, possible to counterfactually say that if one were to make a single, immediately subsequent high precision measurement of $\hat{x}_{\mathrm{f}}$, then the result would typically differ from $\mu_{\mathrm{Xf}}$ by an amount $\sim\Delta_{\mathrm{ef}} x$. It is also possible to counterfactually say, that if one were to make a single, immediately subsequent high precision measurement of $\hat{p}_{\mathrm{f}}$, then the result would typically differ from $\mu_{\mathrm{Pf}}$ by an amount $\sim\Delta_{\mathrm{ef}} p$. There is therefore a well-defined sense in which it may justifiably be said that in recording the meter readings $\mu_{Xf}$, $\mu_{Pf}$, one has simultaneously determined the final values of the position and momentum of the system to accuracy $\pm\Delta_{ef}x$ and $\pm\Delta_{ef}p$ respectively.

The interpretation of the quantities $\Delta_{ei}x$, $\Delta_{ei}p$ is less straightforward. This is because the observables $\hat{x}_i$ and $\hat{\mu}_{Pf}$ do not commute. Nor do the observables $\hat{p}_i$ and $\hat{\mu}_{Xf}$ [see equations (21) in the last section]. This means that the act of making a precise determination of the meter readings $\hat{\mu}_{Xf}$, $\hat{\mu}_{Pf}$ precludes one from making a precise determination of the values of either $\hat{x}_i$ or $\hat{p}_i$. It follows that in the case of the quantities $\Delta_{ei}x$, $\Delta_{ei}p$ we cannot carry through an analysis analogous to the one given in the preceding paragraphs for $\Delta_{ef}x$, $\Delta_{ef}p$.

There is an obvious physical reason why one might expect the concept of retrodictive error to be more problematic than the concept of predictive error. The effect of carrying out a measurement and recording the meter readings is (as we have seen) to put the system into a state such that its *final* position and momentum are confined, with high probability, to a localized region of phase space. However, this is an effect produced by the measurement process itself. If the uncertainties of the initial system state are large, then the *initial* values of the position and momentum will be quite indeterminate. In such a case the concept of retrodictive error does not really make sense. At least, the concept does not make sense if it is defined in anything like the classical manner.

Classically, one thinks of the retrodictive error as the difference between the result of the measurement and the value which the quantity being measured did take before the measurement was carried out. In quantum mechanics, however, the quantity being measured may not have had a well-defined initial value.

Nevertheless, there is at least one situation in which it is possible to attach a meaning to the concept that is similar to the meaning which it has in classical physics. In Section 1 we stated that one of the reasons that an error principle is needed is to justify the assumption (which plays an essential role in experimental physics) that it is normally possible to determine both the position and momentum of a macroscopic object to within a very small percentage error. Suppose that it is a measurement such as this which is in question. Then it will usually be reasonable to assume that the initial system state is a localized wave packet. In other words, the uncertainties $\Delta x_i$, $\Delta p_i$ may be assumed to be small. The purpose of the measurement is to determine the mean values $\bar{x}_i = \langle\psi|\hat{x}_i|\psi\rangle$ and $\bar{p}_i = \langle\psi|\hat{p}_i|\psi\rangle$. If the measurement is retrodictively unbiased

$$\langle\psi \otimes \phi_{ap}|\hat{\mu}_{Xf}|\psi \otimes \phi_{ap}\rangle = \bar{x}_i$$

$$\langle\psi \otimes \phi_{ap}|\hat{\mu}_{Pf}|\psi \otimes \phi_{ap}\rangle = \bar{p}_i$$

In view of equations (24) and (25) we then have

$$\begin{aligned}\langle\psi\otimes\phi_{ap}|(\hat{\mu}_{Xf}-\bar{x}_i)^2\,|\psi\otimes\phi_{ap}\rangle &= (\Delta\mu_{Xf})^2 = (\Delta x_i)^2 + (\Delta_{ei}x)^2 \\ \langle\psi\otimes\phi_{ap}|(\hat{\mu}_{Pf}-\bar{p}_i)^2\,|\psi\otimes\phi_{ap}\rangle &= (\Delta\mu_{Pf})^2 = (\Delta p_i)^2 + (\Delta_{ei}p)^2\end{aligned} \tag{28}$$

It follows that the process determines the values of $\bar{x}_i$, $\bar{p}_i$ up to an uncertainty of $\pm\sqrt{(\Delta x_i)^2 + (\Delta_{ei}x)^2}$ in the determination of $\bar{x}_i$, and $\pm\sqrt{(\Delta p_i)^2 + (\Delta_{ei}p)^2}$ in the determination of $\bar{p}_i$. The quantities $\Delta_{ei}x$, $\Delta_{ei}p$ represent the parts of the total error which arise from the measurement process itself, as opposed to the intrinsic uncertainties of the initial state. In other words, they represent the experimental errors.

If the initial system state is not a localized wave packet, then the classical or ordinary intuitive concept of retrodictive error does not apply. One should realize, however, that this has nothing especially to do with the fact that we are considering simultaneous measurements of position and momentum. Exactly the same problem arises when interpreting the quantity $\Delta x_{\text{measurement}}$ defined by Braginsky and Khalili (1992) for single measurements of position only. It is a simple consequence of the fact that quantum mechanical observables need not take determinate values. This feature of the quantum mechanical theory of measurement is sometimes expressed by saying that we create the value by the act of measuring it.

Although they are then not interpretable as errors in the classical sense, the quantities $\Delta_{ei}x$, $\Delta_{ei}p$ are still defined when the initial system state does not take the form of a localized wave packet. Furthermore, they still play a role in characterizing the "goodness" or "faithfulness" of the measurement. Suppose, for instance, that the initial system state is a superposition of a finite or countable number of well-separated, localized wave packets:

$$|\psi\rangle = \sum_n c_n|\chi_n\rangle$$

In this expression $|\psi\rangle$ and the $|\chi_n\rangle$ are all assumed to be normalized. Define

$$\begin{aligned}\bar{x}_{in} &= \langle\chi_n|\,\hat{x}\,|\chi_n\rangle \\ \bar{p}_{in} &= \langle\chi_n|\,\hat{p}\,|\chi_n\rangle\end{aligned}$$

and

$$l_X = \min_{n\neq m}|\bar{x}_{in} - \bar{x}_{im}|$$

$$l_P = \min_{n\neq m}|\bar{p}_{in} - \bar{p}_{im}|$$

For the sake of simplicity assume that the states $|\chi_n\rangle$ all have the same intrinsic uncertainties $\sigma_X$, $\sigma_P$:

$$\sigma_{\mathrm{X}} = \sqrt{\langle \chi_n | \, (\hat{x} - \bar{x}_{\mathrm{in}})^2 | \, \chi_n \rangle}$$
$$\sigma_{\mathrm{P}} = \sqrt{\langle \chi_n | \, (\hat{p} - \bar{p}_{\mathrm{in}})^2 | \, \chi_n \rangle}$$

for all $n$. The assumption that the wave packets are well separated means that $\sigma_{\mathrm{X}} << l_{\mathrm{X}}$ and $\sigma_{\mathrm{P}} << l_{\mathrm{P}}$. We then have

$$\langle \chi_n | \chi_m \rangle \approx \delta_{nm}$$

and

$$\sum_n |c_n|^2 \approx 1$$

Now surround each point $(\bar{x}_{\mathrm{in}}, \bar{p}_{\mathrm{in}})$ with a region $\mathcal{R}_n$ whose dimensions are large compared with the intrinsic uncertainties $\sigma_{\mathrm{X}}$, $\sigma_{\mathrm{P}}$, but small compared with the minimum separations $l_{\mathrm{X}}$, $l_{\mathrm{P}}$:

$$\mathcal{R}_n = \{(x, p) \in \mathbb{R}^2 \colon |x - \bar{x}_{\mathrm{in}}| < d_{\mathrm{X}}, \, |p - \bar{p}_{\mathrm{in}}| < d_{\mathrm{P}}\}$$

where $\sigma_{\mathrm{X}} << d_{\mathrm{X}} << l_{\mathrm{X}}$ and $\sigma_{\mathrm{P}} << d_{\mathrm{P}} << l_{\mathrm{P}}$. Suppose that we also have $\Delta_{\mathrm{ei}} x << d_{\mathrm{X}}$ and $\Delta_{\mathrm{ei}} p << d_{\mathrm{P}}$. In view of (28), the function $|\langle x, \mu_{\mathrm{X}}, \mu_{\mathrm{P}} | \hat{U} \, | \chi_m \otimes \phi_{\mathrm{ap}} \rangle|^2$ is then concentrated on the set $\mathbb{R} \times \mathcal{R}_m$. Hence

$$\int_{\mathbb{R} \times \mathcal{R}_n} dx \, d\mu_{\mathrm{X}} \, d\mu_{\mathrm{p}} \, |\langle x, \mu_{\mathrm{X}}, \mu_{\mathrm{P}} | \hat{U} \, | \chi_m \otimes \phi_{\mathrm{ap}} \rangle|^2 \approx \delta_{nm}$$

Consequently

$$\int_{\mathbb{R} \times \mathcal{R}_n} dx \, d\mu_{\mathrm{X}} \, d\mu_{\mathrm{P}} \, |\langle x, \mu_{\mathrm{X}}, \mu_{\mathrm{P}} | \hat{U} | \psi \otimes \phi_{\mathrm{ap}} \rangle|^2 \approx |c_n|^2$$

In words: the probability that the final pointer positions will be in the vicinity of the point $(\bar{x}_{in}, \bar{p}_{\mathrm{in}})$ is approximately $|c_n|^2$, provided that the rms errors of retrodiction are sufficiently small.

This result may be regarded as a generalization of the following well-known fact regarding measurements of a single, discrete observable $\hat{A}$. Let $|a\rangle$ be the eigenstate of $\hat{A}$ with eigenvalue $a$, and suppose that the system is in the state

$$|\psi\rangle = \sum_a c_a |a\rangle$$

Suppose that one performs a perfectly precise measurement of $\hat{A}$. Then the probability of recording the value $a$ is $|c_a|^2$. The analogy between this proposition and the result just proved lends some support to the suggestion that processes of the kind described by Arthurs and Kelly may be regarded as simultaneous measurements of noncommuting observables.

Finally, let us consider the interpretation of the quantities $\Delta_d x$, $\Delta_d p$. Suppose that the measurement is both retrodictively and predictively unbiased. By an argument similar to the one leading to equations (28) we find

$$\langle \psi \otimes \phi_{ap} | (\hat{x}_f - \bar{x}_i)^2 | \psi \otimes \phi_{ap} \rangle = (\Delta x_f)^2 = (\Delta x_i)^2 + (\Delta_d x)^2$$
$$\langle \psi \otimes \phi_{ap} | (\hat{p}_f - \bar{p}_i)^2 | \psi \otimes \phi_{ap} \rangle = (\Delta p_f)^2 = (\Delta p_i)^2 + (\Delta_d p)^2$$

where $\bar{x}_i$, $\bar{p}_i$ are the expectation values of $\hat{x}_i$, $\hat{p}_i$, as before. The effect of the measurement process on the system state is to leave the expectation value of $\hat{x}$ (respectively $\hat{p}$) unchanged, while increasing the variance by an amount $(\Delta_d x)^2$ [respectively, $(\Delta_d p)^2$]. There is thus a well-defined sense in which the quantities $\Delta_d x$, $\Delta_d p$ provide a numerical indication of the extent to which the measurement disturbs the state of the system.

## 6. CONCLUSION

Does quantum mechanics allow for the existence of simultaneous measurements of position and momentum? We can see no clear objection to the use of the term "measurement" to refer to the kind of process described by Arthurs and Kelly. However, it must be admitted that insofar as the question at issue is one of nomenclature, it probably does not have a once-and-for-all right answer. Such questions are, in the end, a matter of taste.

What is not a matter of taste is the fact that processes of the kind considered are of some importance in the field of quantum optics. This is true irrespective of the name by which one chooses to describe them. If the quantities introduced in this paper are to be of any interest, they must be justified in the same way, in terms of their usefulness. Braginsky and Khalili have shown that the relationship they derive is a useful tool in the analysis of single measurements of $\hat{x}$ or $\hat{p}$ separately. It seems not unreasonable to suppose that the relationships derived in this paper may be no less useful in the analysis of simultaneous measurements of $\hat{x}$ and $\hat{p}$ together. At the least, they seem worthy of further investigation.

## APPENDIX

In Section 3 we rely on a proposition which forms the basis of the argument in both Arthurs and Kelly (1965) and Arthurs and Goodman (1988). However, in neither case do the authors actually prove this proposition. Since it is not entirely obvious, we give the proof here.

*Proposition.* Let $\mathscr{H}_1$, $\mathscr{H}_2$ be two Hilbert spaces and let $\hat{A}$ be a (possibly unbounded) linear operator defined on the product space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Let $\mathscr{D}$

$\subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$ be the domain of $\hat{A}$. Let $|\phi\rangle$ be a fixed vector $\in \mathcal{H}_2$. Suppose that $\mathcal{H}_1 \otimes |\phi\rangle \subseteq \mathcal{D}$, and suppose also that

$$\langle \psi \otimes \phi | \hat{A} | \psi \otimes \phi \rangle = 0 \tag{A1}$$

for all $|\psi\rangle \in \mathcal{H}_1$. Then

$$\langle \psi \otimes \phi | \hat{A} | \psi' \otimes \phi \rangle = 0$$

for all $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_1$.

*Proof.* The result is proved in essentially the same way as (for example) Proposition 2.4.3 in Kadison and Ringrose (1983). Given arbitrary $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_1$ we have the identity

$$\begin{aligned} &\langle \psi \otimes \phi | \hat{A} | \psi' \otimes \phi \rangle \\ &= \frac{1}{4} (\langle (\psi + \psi') \otimes \phi | \hat{A} | (\psi + \psi') \otimes \phi \rangle \\ &\quad - \langle (\psi - \psi') \otimes \phi | \hat{A} | (\psi - \psi') \otimes \psi \rangle \\ &\quad - i \langle (\psi + i\psi') \otimes \phi | \hat{A} | (\psi + i\psi') \otimes \phi \rangle \\ &\quad + i \langle (\psi - i\psi') \otimes \phi | \hat{A} | (\psi - i\psi') \otimes \phi \rangle) \end{aligned}$$

Using equation (A1), we deduce

$$\langle \psi \otimes \phi | \hat{A} | \psi' \otimes \phi \rangle = 0$$

## REFERENCES


Arthurs, E., and Goodman, M. S. (1988). *Physical Review Letters*, **60**, 2447.

Arthurs, E., and Kelly, J. L., Jr. (1965). *Bell System Technical Journal*, **44**, 725.

Ballentine, L. F. (1970). *Reviews of Modern Physics*, **42**, 358.

Bohm, D. (1951). *Quantum Theory*, Prentice-Hall, New York.

Braginsky, V. B., and Khalili, F. Ya (1992). *Quantum Measurement*, K. S. Thorne, ed., Cambridge University Press, Cambridge.

Braunstein, S. L., Caves, C. M., and Milburn, G. J. (1991). *Physical Review A*, **43**, 1153.

Busch, P., and Lahti, P. J. (1984). *Physical Review D*, **29**, 1634.

de Muynck, W. M., De Baere, W., and Martens, H. (1994). *Foundations of Physics*, **24**, 1589.

Heisenberg, W. (1927). *Zeitschrift für Physik*, **43**, 172, [Reprinted in Wheeler, J. A., and Zurek, W. H., eds. (1983). *Quantum Theory and Measurement*, Princeton University Press, Princeton, New Jersey.]

Heisenberg, W. (1930). *The Physical Principles of the Quantum Theory*, University of Chicago Press, Chicago.

Hilgevoord, J., and Uffink, J. (1990). In *Sixty-Two Years of Uncertainty*, A. I. Miller, ed., Plenum Press, New York.

Holevo, A. S. (1982). *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam.

Kadison, R. V., and Ringrose, J. R. (1983). *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York.

Kennard, E. H. (1927). *Zeitschrift für Physik*, **44**, 326.

Leonhardt, U., and Paul, H. (1993). *Journal of Modern Optics*, **40**, 1745.

Leonhardt, U., Böhmer, B., and Paul, H. (1995). *Optics Communications*, **119**, 296.

Martens, H., and de Muynck, W. M. (1992). *Journal of Physics A*, **25**, 4887.

Prugovečki, E. (1973). *Foundations of Physics*, **3**, 3.

Prugovečki, E. (1975). *Foundations of Physics*, **5**, 557.

Prugovečki, E. (1984). *Stochastic Quantum Mechanics and Quantum Spacetime*, Reidel, Dordrecht.

Raymer, M. G. (1994). *American Journal of Physics*, **62**, 986.

Stenholm, S. (1992). *Annals of Physics*, **218**, 233.

Törma, P., Stenholm, S., and Jex, I. (1995). *Physical Review A*, **52**, 4812.

Uffink, J. (1994). *International Journal of Theoretical Physics*, **33**, 199.

Wódkiewicz, K. (1987). *Physics Letters A*, **124**, 207.